Exercise 13

Convert each of the following Volterra integral equation in 9–16 to an equivalent IVP:

$$u(x) = 1 - \cos x + 2\int_0^x (x-t)^2 u(t)\, dt$$

Solution

Differentiate both sides with respect to x.

$$u'(x) = \sin x + 2\frac{d}{dx}\int_0^x (x-t)^2 u(t)\, dt$$

Use the Leibniz rule to differentiate the integral.

$$\begin{aligned} &= \sin x + 2\left[\int_0^x \frac{\partial}{\partial x}(x-t)^2 u(t)\, dt + (0)^2 u(x)\cdot 1 - (x)^2 u(0)\cdot 0\right] \\ &= \sin x + 2\left[\int_0^x 2(x-t)u(t)\, dt\right] \\ &= \sin x + 4\int_0^x (x-t)u(t)\, dt \end{aligned}$$

Differentiate both sides with respect to x again.

$$\begin{aligned} u''(x) &= \cos x + 4\frac{d}{dx}\int_0^x (x-t)u(t)\, dt \\ &= \cos x + 4\left[\int_0^x \frac{\partial}{\partial x}(x-t)u(t)\, dt + (0)u(x)\cdot 1 - (x)u(0)\cdot 0\right] \\ &= \cos x + 4\int_0^x u(t)\, dt \end{aligned}$$

Differentiate both sides with respect to x again.

$$\begin{aligned} u'''(x) &= -\sin x + 4\frac{d}{dx}\int_0^x u(t)\, dt \\ &= -\sin x + 4u(x) \\ u''' - 4u &= -\sin x \end{aligned}$$

The initial conditions to this ODE are found by plugging in $x = 0$ into the original integral equation,

$$u(0) = 1 - \cos 0 + 2\int_0^0 (0-t)^2 u(t)\, dt = 0,$$

and the formula for u',

$$u'(0) = \sin 0 + 4\int_0^0 (0-t)u(t)\, dt = 0,$$

and the formula for u'',

$$u''(0) = \cos 0 + 4\int_0^0 u(t)\, dt = 1.$$

Therefore, the equivalent IVP is

$$u''' - 4u = -\sin x,\ u(0) = 0,\ u'(0) = 0,\ u''(0) = 1.$$